Complex Analysis (for Physics)
Final Exam (with solutions)
Exam Date: January 26, 2023 (11:45 – 13:45)

university of groningen

1) Consider the complex function

$$F(z) = z + (z^2 + \overline{z}^2) + i|z|^2$$

a) Determine the point(s) at which $F(z)$ is differentiable.
b) Compute the derivative $F'(z)$ at the point(s) where it exists.

Solution. a) We start by identifying the real and imaginary parts of the function, i.e.

$$u(x, y) = \operatorname{Re} F(z) = \operatorname{Re}\{z + (z^2 + \overline{z}^2) + i|z|^2\} = x + 2(x^2 - y^2)$$

and

$$v(x, y) = \operatorname{Im} F(z) = \operatorname{Im}\{z + (z^2 + \overline{z}^2) + i|z|^2\} = y + (x^2 + y^2)$$

This shows that the function is defined everywhere with continuously differentiable real and imaginary parts, therefore $F(z)$ is differentiable exactly at the points $z_0 = x_0 + iy_0 \in \mathbb{C}$ where the Cauchy-Riemann equations are satisfied (cf. Theorem 5 of Section 2.4). The Cauchy-Riemann equations read

$$\left.\begin{array}{rcl} \dfrac{\partial u}{\partial x} & = & \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} & = & -\dfrac{\partial v}{\partial x} \end{array}\right\} \iff \left\{\begin{array}{rcl} 1 + 4x & = & 1 + 2y \\ -4y & = & -2x \end{array}\right.$$

The first equation implies that $y = 2x$ whereas the second equation implies that $4y = 2x$. Taking the difference of these two equations yields $3y = 0$, i.e. $y = 0$ and then $x = 0$ as well. This means that the origin $z_0 = 0 + i0 = 0$ is the only point at which the Cauchy-Riemann equations are satisfied hence the only point where the derivative of $F(z)$ exists.

b) Given that the derivative exists at z_0, we have

$$F'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial u}{\partial y}(x_0, y_0).$$

Based on part a), we see that

$$F'(0) = [(1 + 4x) + i(-4y)] \,|_{(x,y)=(0,0)} = 1$$

2) Consider the complex function

$$g(z) = \frac{3}{(z-2)(z+1)}$$

a) Find its Taylor series and the circle of convergence around 0.
b) Find its Laurent series expansion in the domain $|z| > 2$.
c) Determine its singularities (with type and order specified).

Solution. a) Using partial fraction decomposition we find that

$$g(z) = \frac{1}{z-2} + \frac{-1}{z+1}$$

which is suitable for applying the geometric series formula. Namely, the first term can be written as

$$\frac{1}{z-2} = -\frac{1}{2}\frac{1}{1-\frac{z}{2}} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \quad \text{if } \left|\frac{z}{2}\right| < 1, \text{ i.e. } |z| < 2$$

and the second term can be expressed as

$$\frac{-1}{z+1} = -\frac{1}{1-(-z)} = -\sum_{n=0}^{\infty}(-z)^n \quad \text{if } |-z| < 1, \text{ i.e. } |z| < 1.$$

Therefore for points inside the unit circle $|z| < 1$ we have the following Taylor series expansion of $g(z)$ around 0:

$$g(z) = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \sum_{n=0}^{\infty}(-z)^n = \sum_{n=0}^{\infty}\left((-1)^{n+1} - \frac{1}{2^{n+1}}\right)z^n.$$

b) The choice of domain $|z| > 2$ suggests a different way of expressing the terms in the partial fraction decomposition seen in part a). Namely, we have

$$\frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\frac{2}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n \quad \text{if } \left|\frac{2}{z}\right| < 1, \text{ i.e. } |z| > 2$$

and

$$\frac{-1}{z+1} = -\frac{1}{z}\frac{1}{1-\left(-\frac{1}{z}\right)} = -\frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n \quad \text{if } \left|-\frac{1}{z}\right| < 1, \text{ i.e. } |z| > 1.$$

Thus the Laurent series of $g(z)$ in the domain $|z| > 2$ reads

$$g(z) = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n = \sum_{n=0}^{\infty}[2^n - (-1)^n]z^{-(n+1)} = \sum_{n=1}^{\infty}[2^n - (-1)^n]z^{-(n+1)}.$$

Note that the coefficient of $\frac{1}{z}$ (i.e. $n = 0$ term) is zero.

c) Being a rational function, the singularities of $g(z)$ in $\mathbb{C}$ are poles located at the zeros of the denominator (with matching orders). Therefore the singularities of $g(z)$ are simple poles located at $z = 2$ and $z = -1$.

3) Compute the following complex integral

$$\oint_{\Gamma}(2iz - 3\overline{z})\,dz$$

where Γ is the positively oriented contour consisting of the interval $[-1, 1]$ and the upper semicircle of radius 1 centered at 0.

Solution. The contour is the union of a line segment C_1 and a semicircle C_2 which can be parameterized by the functions $z_1(t) = t$, $-1 \le t \le 1$ and $z_2(t) = e^{it}$, $0 \le t \le \pi$, respectively. Note that we have $z_1'(t) = 1$ and $z_2'(t) = ie^{it}$. Using the additivity of complex integrals we get

$$\begin{aligned}
\oint_{\Gamma}(2iz - 3\overline{z})\,dz &= \int_{C_1}(2iz - 3\overline{z})\,dz + \int_{C_2}(2iz - 3\overline{z})\,dz \\
&= \int_{-1}^{1}(2it - 3t)\,dt + \int_{0}^{\pi}(2ie^{it} - 3e^{-it})ie^{it}\,dt \\
&= [it^2 - \tfrac{3}{2}t^2]_{-1}^{1} + [ie^{2it} - 3it]_0^{\pi} \\
&= -3\pi i.
\end{aligned}$$

4) Evaluate the following improper integral

$$\int_0^{\infty}\frac{x^2\cos(2x)}{x^4 + 6x^2 + 9}\,dx$$

Solution. Since the integrand is an even function, we have

$$\int_0^\infty \frac{x^2\cos(2x)}{x^4+6x^2+9}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2\cos(2x)}{x^4+6x^2+9}\,dx.$$

Let us consider the complex function

$$f(z) = \frac{z^2 e^{2iz}}{z^4+6z^2+9} = \frac{z^2}{(z^2+3)^2}e^{2iz}.$$

By Euler's formula, it is clear that we have

$$\operatorname{Re} f(z) = \frac{x^2\cos(2x)}{x^4+6x^2+9}.$$

The function $f(z)$ has two double poles at $z_1 = i\sqrt{3}$ and $z_2 = -i\sqrt{3}$. Let Γ_R denote the positively oriented closed contour consisting of the interval $[-R,R]$ along the real axis and the upper semicircle $C_R^+(0)$ of radius R centered at 0. If $R > \sqrt{3}$, then z_1 is inside Γ_R whereas z_2 is not enclosed by Γ_R for any $R > 0$. Therefore, by the Residue Theorem, we have

$$\oint_{\Gamma_R} f(z)\,dz = 2\pi i \operatorname{Res}(f, z_1), \quad \text{if } R > \sqrt{3}.$$

Since z_1 is a pole of order 2, the residue is found via

$$\operatorname{Res}(f,z_1) = \lim_{z\to z_1}\left((z-z_1)^2 f(z)\right)' = \lim_{z\to z_1}\left(\frac{z^2e^{2iz}}{(z+i\sqrt{3})^2}\right)' = \frac{6-\sqrt{3}}{12}e^{-2\sqrt{3}}i.$$

Thus

$$\oint_{\Gamma_R} f(z)\,dz = 2\pi i\operatorname{Res}(f,z_1) = \left(\frac{1}{2\sqrt{3}}-1\right)\pi e^{-2\sqrt{3}}.$$

We also have

$$\oint_{\Gamma_R} f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R^+(0)} f(z)\,dz.$$

By combining the two expressions for the contour integral we obtain

$$\int_{-R}^{R} f(x)\,dx = \left(\frac{1}{2\sqrt{3}}-1\right)\pi e^{-2\sqrt{3}} - \int_{C_R^+(0)} f(z)\,dz$$

and thus

$$\int_{-\infty}^{\infty} f(x)\,dx = \left(\frac{1}{2\sqrt{3}}-1\right)\pi e^{-2\sqrt{3}} - \lim_{R\to\infty}\int_{C_R^+(0)} f(z)\,dz$$

The last term vanishes due to Jordan's lemma as we have $\deg(z^4+6z^2+9) \geq 1+\deg(z^2)$ and we are left with

$$\int_{-\infty}^{\infty} f(x)\,dx = \left(\frac{1}{2\sqrt{3}}-1\right)\pi e^{-2\sqrt{3}}.$$

Thus we have computed the improper integral in question and found that

$$\int_0^\infty \frac{x^2\cos(2x)}{x^4+6x^2+9}\,dx = \frac{1}{2}\operatorname{Re}\int_{-\infty}^{\infty} f(x)\,dx = \left(\frac{1}{2\sqrt{3}}-1\right)\frac{\pi}{2}e^{-2\sqrt{3}}.$$

5) Suppose that $f(z)$ is analytic on the closed disc $|z| \leq 1$ and satisfies $|f(z)| < 1$ along the boundary circle $|z| = 1$. Show that $f(z)$ has exactly one *fixed point* inside the disc, i.e. the equation $f(z) = z$ has exactly one solution (counting multiplicity) in $|z| < 1$.

Solution. For the entire function $h(z) = -z$ we have $|f(z)| < 1 = |z| = |-z| = |h(z)|$ along the unit circle $|z| = 1$. Therefore, by Rouché's Theorem, $f(z) + h(z) = f(z) - z$ and $h(z) = -z$ have the same number of zeros inside the unit disc. The function $h(z) = -z$ clearly has exactly one zero (of multiplicity 1 at $z = 0$), therefore $f(z) - z$ also has exactly one zero satisfying $|z| < 1$, i.e. $f(z) - z = 0$ has exactly one solution in $|z| < 1$. This concludes the proof.